ITCS 2026

Debordering Closure Results in Determinantal and Pfaffian Ideals

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Algebraic Circuits with Oracles

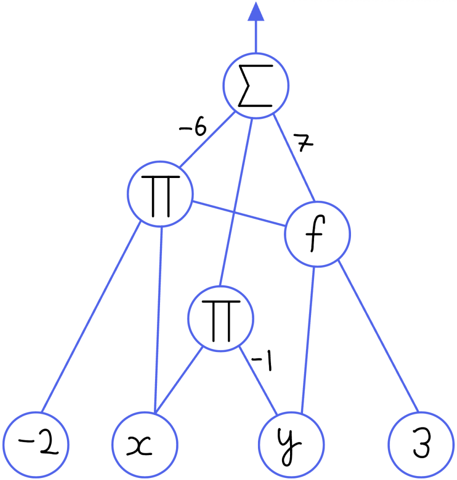


Figure: $-6(-2x \cdot f(y,3)) - xy + 7f(y,3)$

We have two measures of complexity:

- *Size:* The number of edges
- *Depth:* The length of the longest input-output path

Circuit Complexity for Ideals

In algebraic complexity, we are interested in characterizing the *circuit complexity* of some family of polynomials.

Definition

Fix some polynomials $g_1(\overline{x}), \ldots, g_k(\overline{x}) \in \mathbb{F}[\overline{x}]$.
The *ideal generated by* $g_1(\overline{x}), \ldots, g_k(\overline{x})$ is the set of combinations

$$\langle\, g_1, \ldots, g_k \,\rangle = \left\{ \sum_{i=1}^{k} h_i(\overline{x}) \cdot g_i(\overline{x}) \quad \middle| \quad h_i(\overline{x}) \in \mathbb{F}[\overline{x}] \right\}.$$

Question

Suppose $f \in \langle\, g_1, \ldots, g_k \,\rangle$. How does the complexity of f compare to the complexity of the generators $g_1, \ldots, g_k$?

Principal Ideals

Example

The *principal ideals* are generated by a single polynomial g.
If $f \in \langle g \rangle$, then g is a *factor* of f.

Question

Suppose $f \in \langle g \rangle$. Does g have a small f-oracle circuit?

Principal Ideals

Conjecture ([Bür00, Conjecture 8.3])

If g is a factor of f, $\text{size}(g) \leq \text{poly}(\text{size}(f), \deg(f))$.

Theorem ([Bür04, Theorem 1.3])

Over fields of characteristic 0, g can be *border computed* by a circuit of size $\text{poly}(\text{size}(f), \deg(f))$.

By border computation, we mean the circuit computes the following:

$$g(\overline{x}) + \varepsilon^q \tilde{g}(\overline{x}, \varepsilon) \in \mathbb{F}(\varepsilon)[\overline{x}], \qquad q \geq 1.$$

Question

Can we *deborder* this result, that is can we remove this ε approximation?

Closure Results in Determinantal Ideals

Example

Consider an $n \times m$ matrix $X = (x_{i,j})$ of variables. Let $I^{\det}_{n,m,r}$ be the *determinantal ideal* generated by the $r \times r$ minors of X.

Conjecture ([Gro20, Conjecture 6.3])

Let $f \in I^{\det}_{n,n,n/2}$ be a nonzero polynomial. Then there is a constant depth f-oracle circuit of size $\text{poly}(n)$ that computes the $t \times t$ determinant for some $t = n^{\Theta(1)}$.

Closure Results in Determinantal Ideals

Theorem ([AF22, Theorem 1.1])

Let $f \in I^{\text{det}}_{n,m,r}$ be a nonzero polynomial. Then there is a depth-three f-oracle circuit of size $O\left(n^2m^2\right)$ that *border computes* the $t \times t$ determinant for some $t = \Theta\left(r^{1/3}\right)$.

Question

Can we *deborder* this result, that is can we remove this ε approximation?

Closure Results in Determinantal Ideals

Theorem ([DG25, Theorem 1.5])

Let $f \in I_{n,m,r}^{\det}$ be a nonzero polynomial. Then there is a depth-three f-oracle circuit of size $\mathrm{poly}(n, m, \deg(f))$ that *exactly computes* the $t \times t$ determinant for some $t = \Theta\left(r^{1/3}\right)$.

Main Tools:

- We use *Straightening Laws* from Invariant Theory to express $f(\overline{x})$ in a standard basis indexed by combinatorial objects, and leverage the combinatorics to talk about specific terms.
- To get a circuit for a specific basis term, we use *Homogenization* as well as the *Isolation Lemma*.

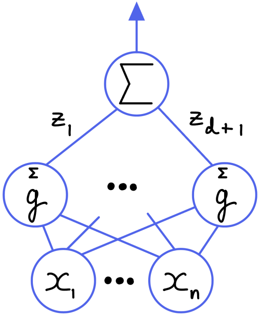
Homogenization

Definition

Consider a degree d polynomial $g(\overline{x}, t) = \sum_{i=0}^{d} \text{coeff}_{t^i}(g) t^i$.

Lemma (Folklore)

Say g is computed by a size s, depth Δ f-oracle circuit with top Σ-gate.



Then, we can compute $\text{coeff}_{t^i}(g)$ by a size $O(ds)$, depth Δ f-oracle circuit.

Issues with Homogenization

If a circuit border computes $g(\overline{x})$, then it computes

$$g(\overline{x}) + \varepsilon^q \tilde{g}(\overline{x}, \varepsilon)$$

for some $q \geq 1$.

Idea: Homogenize with respect to ε.

Problem: q can be *arbitrarily large*

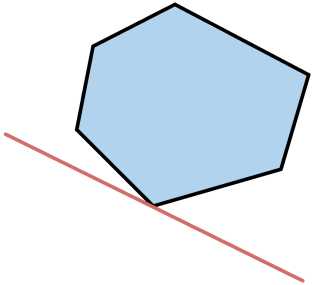
$\implies$ Homogenization gives *large circuit*.

Isolation Lemma

In our proof, we have a specific monomial in $g(\overline{x})$ we want to *isolate*.

Lemma ([KS01, Lemma 4])

Linear programs with *random* cost functions will have a unique minimum.



Moreover, if the linear equations have bounded integer coefficients, then evaluation at *small, random* values has a unique minimum.

Isolating Monomials

Lemma ([DG25, Corollary 2.27])

Consider a polynomial $g(x_1, \ldots, x_\ell)$ such that the individual degree of each x_i in g is at most K:

$$g(x_1, \ldots, x_\ell) = \sum_{\overline{e} \in \mathcal{L}} c_{\overline{e}} x_1^{e_1} \cdots x_\ell^{e_\ell}.$$

Randomly choose $z_1, \ldots, z_\ell$ and define a morphism

$$x_i \mapsto w^{z_i}, \qquad g \mapsto \sum_{\overline{e} \in \mathcal{L}} c_{\overline{e}} \cdot w^{\sum_{i=1}^{\ell} e_i z_i}.$$

The Isolation Lemma shows that the $z_1, \ldots, z_\ell$ can be choose to be small $\implies$ unique lowest $\deg_w$-term $\implies$ homogenization w.r.t w is small.

Thank You!

If I am to speak ten minutes, I need a week for preparation; if fifteen minutes, three days; if half an hour, two days; if an hour, I am ready now.

— Woodrow Wilson

Slides can be found on my site anakin.phd

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